Path integration in the field of a topological defect: the case of dispiration

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The motion of a particle in the field of dispiration (due to a wedge disclination and a screw dislocation) is studied by path integration. By formulating a $SO(2) \times T(1)$ gauge theory, first, we derive the metric, curvature, and torsion of the medium of dispiration. Then we carry out explicitly path integration for the propagator of a particle moving in the non-Euclidean medium under the influence of a scalar potential and a vector potential. We obtain also the winding number representation of the propagator by taking the non-trivial topological structure of the medium into account. We extract the energy spectrum and the eigenfunctions from the propagator. Finally we make some remarks for special cases. Particularly, paying attention to the difference between the result of the path integration and the solution of Schrödinger's equation in the case of disclination, we suggest that the Schrödinger equation may have to be modified by a curvature term.

PACS: 61.72Lk - Quantum mechanics, linear defects: dislocations, disclinations.

Keywords - Path integral

1 Introduction

In recent years quantum effects on particle propagation in a field of topological defects have attracted considerable attention (see, e.g., [1, 2, 3]; see also a recent review article [4]). Although the notion of "defects" in physics was originally associated with crystalline irregularities, it has been extended to more general topological structures such as entangled polymers, liquid crystals, vortices, anyons, global monopoles, cosmic strings, domain walls, etc. In 1950's, from the structuralogical aspect, Kondo *et al.*[5] extensively studied unified geometrical treatments of various subjects including elastic and plastic media, relativity, network systems, and information theory. In particular, Kondo [6] related dislocations to Cartan's torsion in the medium. Since then the relation between dislocation theory and non-Riemannian geometry has been well established. More recent approaches to defect problems are gauge theories similar to those in particle physics [7, 8, 9, 10]. In 1978, Kawamura [11] pointed out that a screw dislocation in a crystal produces an Aharonov-Bohm type effect in particle scattering. The Aharonov-Bohm effect [12] is usually understood as a topological effect [13, 14, 15, 16]. Since standard approaches to particle-defect interaction problems constitute of solving relevant (local) differential equations, the role of topology becomes often obscure.

In the present paper, we analyze the quantum behavior of a particle in the vicinity of a topological defect by the path integral method. Specifically we carry out path integration for a particle moving around a dispiration[1]. The dispiration we consider is a composite structure of a screw dislocation and a wedge disclination with a common defect line. In Sec. 2, we study the geometrical and topological properties of the dispiration by using the $SO(2) \times T(1)$ gauge theory. For the medium $\mathcal{M}$ of the dispiration, we derive the squared line element,

$$ds^2 = dr^2 + \sigma^2 r^2 d\theta^2 + (dz + \beta d\theta)^2, \tag{1}$$

where β and σ are the parameters directly related to the Burgers vector of dislocation and the Frank vector of disclination, respectively. The medium has a non-Euclidean structure with singular torsion and curvature along the dispiration line.

In Sec. 3, we calculate in detail the path integral to obtain the propagator (Feynman kernel) for a particle in the field of the dispiration characterized by the line element (1) under the influence of a scalar potential and a vector potential. From the propagator so obtained, we extract the energy spectrum and the energy eigenfunctions. In Sec. 4, converting the propagator in the partial wave expansion into that in the winding number representation, we show that the effect due to the multiply connected structure of the medium is taken into account. The energy eigenfunctions and the energy spectrum are extracted from the propagator in Sec. 5. Sec.6 is devoted to interpretation of the energy spectrum. In particular, we discuss the difference between the results from the Schrödinger equation with no curvature term and those from the path integral for the case of the conical space. Our path integral calculation suggests that the standard Schrödinger equation may have to be modified by a curvature term in order for two approaches to be consistent.

2 Gauge formulation of dispiration

The dispiration under consideration is a combined structure of a screw dislocation along the z-axis and a wedge disclination about the same z-axis. See Figs. 1-2. The gauge theoretical approach to dislocation and disclination has been extensively discussed in the literature [7, 8, 9, 10]. Here we wish to present a $SO(2) \times T(1)$ gauge approach to the dispiration developed along the line similar to the formulation by Puntigam and Soleng [10].

The $SO(2) \times T(1)$ gauge transformation:– In the gauge theoretical treatment, a deformation of an elastic medium in three dimensions is described by a local coordinate transformation consisting of a rotation and a translation,

$$\mathbf{x}' = \boldsymbol{\rho}(\mathbf{x}) \cdot \mathbf{x} + \boldsymbol{\tau}(\mathbf{x}), \tag{2}$$

where $\boldsymbol{\rho} \in SO(2)$, $\boldsymbol{\tau} \in T(1)$, and $\mathbf{x} = \{x, y, z\}$ is the position vector of the undeformed three-dimensional medium.

As is well-known, an axial wedge disclination can be created by the so-called Volterra process: That is, by (a) removing a wedge shaped portion of an angle $\gamma \in [0, 2\pi)$ from the medium and pasting the open walls together, or (b) inserting an extra wedge-shaped portion with $\gamma \in [-2\pi, 0)$ into the medium. Note that the deficit angle γ is chosen to be positive for the case (a) and negative for the case (b). See Fig. 1 for the case (a). The wedge disclination about the z-axis is a rotational deformation obtained by gauging $SO(2)$:

$$\mathbf{x}' = \boldsymbol{\rho}(\theta) \cdot \mathbf{x} \tag{3}$$

with the rotation matrix,

$$\boldsymbol{\rho}(\theta) = \begin{pmatrix} \cos(\gamma\theta/2\pi) & -\sin(\gamma\theta/2\pi) & 0 \\ \sin(\gamma\theta/2\pi) & \cos(\gamma\theta/2\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

[1]The Volterra process of forming a dislocation and a disclination involves a translation and a rotation, respectively. A *dispiration* is a defect formed by a translation and a rotation at the same time [17]. See Fig. 1 and Fig. 2.

Here $\theta = \tan^{-1}(y/x) \in [0, 2\pi)$.

A screw dislocation lying along the z-axis (i.e., having a constant Burgers vector pointing the z-direction) as shown in Fig. 2 is a translational deformation obtained by gauging the z-translational group $T(1)$:

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{\tau}(\theta) \tag{5}$$

with the angle-dependent z-translation vector

$$\boldsymbol{\tau}(\theta) = -\frac{b\theta}{2\pi}\mathbf{e}_z \tag{6}$$

where b is a translation parameter and $\theta = \tan^{-1}(y/x)$ as before.

The dispiration comprised of such a wedge disclination and a screw dislocation is described by the combined coordinate transformation,

$$\mathbf{x}' = \boldsymbol{\rho}(\theta)\cdot\mathbf{x} + \boldsymbol{\tau}(\theta) \tag{7}$$

where $\boldsymbol{\rho}(\theta)$ and $\boldsymbol{\tau}(\theta)$ are given by (4) and (6), respectively.

Gauge connections:– The standard Yang-Mills theory localizes with respect to the external space the gauge group which acts homogeneously in the internal space and introduces the gauge potential or the connection $\mathbf{\Gamma}$ that transforms under the group action $\mathcal{G}$ as

$$\mathbf{\Gamma}' = \mathcal{G}\mathbf{\Gamma}\mathcal{G}^{-1} - d\mathcal{G}\,\mathcal{G}^{-1}.$$

What we wish to formulate for the dispiration is a $SO(2)\times T(1)$ gauge approach which differs in character from the Yang-Mills theory; the gauge group acts inhomogeneously on the (internal) coordinates $\mathbf{x}$ that is soldered locally to the (external) coordinates of the medium of dispiration. It is more appropriate for us to follow the procedures used in constructing a Poincaré gauge theory [10, 18] or an affine gauge theory [19] for gravity.

First we write the group action (7) in the matrix form,

$$\mathcal{G}\bar{\mathbf{x}} = \begin{pmatrix} \boldsymbol{\rho} & \boldsymbol{\tau} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\rho}\cdot\mathbf{x} + \boldsymbol{\tau} \\ 1 \end{pmatrix}.$$

Then we define the connection $\mathbf{\Gamma}$ as

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{\Gamma}^{(R)} & \mathbf{\Gamma}^{(T)} \\ 0 & 0 \end{pmatrix}$$

where $\mathbf{\Gamma}^{(R)}$ and $\mathbf{\Gamma}^{(T)}$ are the rotational connection and the translational connection, respectively. In order for $\mathbf{\Gamma}$ to behave like a connection in the Yang-Mills theory, $\mathbf{\Gamma}^{(R)}$ and $\mathbf{\Gamma}^{(T)}$ must transform as

$$\mathbf{\Gamma}^{(R)\prime} = \boldsymbol{\rho}\mathbf{\Gamma}^{(R)}\boldsymbol{\rho}^{-1} - (d\boldsymbol{\rho})\boldsymbol{\rho}^{-1}$$

and

$$\mathbf{\Gamma}^{(T)\prime} = \boldsymbol{\rho}\mathbf{\Gamma}^{(R)} - d\boldsymbol{\tau} - \left[\boldsymbol{\rho}\mathbf{\Gamma}^{(R)}\boldsymbol{\rho}^{-1} - (d\boldsymbol{\rho})\,\boldsymbol{\rho}^{-1}\right]\boldsymbol{\tau}.$$

Evidently the rotational connection is a $SO(2)$-valued one-form, and the translational connection is a $\mathbf{R}$-valued connection one-form.

Now we construct the solder form that locally connects the global gauge coordinates $\mathbf{x}(0)$ to the coordinates $\mathbf{x}(\theta)$ of the dispiration medium as

$$\boldsymbol{\omega} = d\mathbf{x} + \mathbf{\Gamma}^{(R)}\cdot\mathbf{x} + \mathbf{\Gamma}^{(T)} \tag{8}$$

which is a vector valued one-form.[2] It transforms as $\boldsymbol{\omega}' = \boldsymbol{\rho}\boldsymbol{\omega}$, leaving $g = \boldsymbol{\omega}^T \cdot \boldsymbol{\omega}$ invariant. Since we create the dispiration by a gauge transformation in flat space, we choose a connection that vanishes at $\theta = 0$. Then the corresponding rotation and translation connections are given, respectively, by

$$\boldsymbol{\Gamma}^{(R)} = \boldsymbol{\rho}\, d\boldsymbol{\rho}^{-1} \tag{9}$$

and

$$\boldsymbol{\Gamma}^{(T)} = -d\boldsymbol{\tau}. \tag{10}$$

The rotational connection (9) and the translational connection (10) can be easily calculated by using the rotation matrix (4) and the translation vector (6). The differential of the rotation matrix (4) is

$$d\boldsymbol{\rho}^{-1}(\theta) = d\boldsymbol{\rho}(-\theta) = -\frac{\gamma}{2\pi}\, d\theta \begin{pmatrix} \sin(\gamma\theta/2\pi) & -\cos(\gamma\theta/2\pi) & 0 \\ \cos(\gamma\theta/2\pi) & \sin(\gamma\theta/2\pi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

where

$$d\theta = \frac{1}{r^2}(x\, dy - y\, dx)\ , \quad r^2 = x^2 + y^2. \tag{12}$$

This and the rotation matrix (4) together lead us to the rotational connection,

$$\boldsymbol{\Gamma}^{(R)} = \boldsymbol{\rho}(\theta)\, d\boldsymbol{\rho}^{-1}(\theta) = \frac{\gamma}{2\pi}\, \mathbf{m}\, d\theta \tag{13}$$

where

$$\mathbf{m} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

The translational connection is found in a simple form by differentiating the translation vector (6),

$$\boldsymbol{\Gamma}^{(T)} = -d\boldsymbol{\tau} = \frac{b}{2\pi}\, d\theta\, \mathbf{e}_z. \tag{15}$$

Substitution of (13) and (15) into (8) yields

$$\boldsymbol{\omega} = d\mathbf{x} + \frac{\gamma}{2\pi}\, d\theta\, \mathbf{m} \cdot \mathbf{x} + \frac{b}{2\pi}\, d\theta\, \mathbf{e}_z = \begin{pmatrix} dx & + & (\gamma/2\pi) y\, d\theta \\ dy & - & (\gamma/2\pi) x\, d\theta \\ dz & + & (b/2\pi)\, d\theta \end{pmatrix}. \tag{16}$$

Utilizing this solder form as the coframe we can determine the squared line element $ds^2 = g_{ij}\, dx^i \otimes dx^j$ in the medium of the dispiration,

$$ds^2 = \delta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta = dr^2 + \sigma^2\, r^2\, d\theta^2 + (dz + \beta d\theta)^2, \tag{17}$$

where

$$\sigma = 1 - \frac{\gamma}{2\pi}, \qquad \beta = \frac{b}{2\pi}. \tag{18}$$

Evidently $\sigma \in (0, 1]$ when $\gamma \in [0, 2\pi)$, and $\sigma \in (1, 2]$ when $\gamma \in [-2\pi, 0)$.

[2]It is possible to derive this solder form by starting with the homogeneous group $SO(4)$ and by applying the group contraction to reduce it to $SO(3) \times T(1)$. The solder form may be obtained as a limiting form of a component of the connection. For instance, the de Sitter gauge theory can be contracted to the Poincaré gauge theory, so that the solder form is obtained from the gauge potential as the vierbeine [20]. This contraction scheme does not work for the affine gauge theory [19].

In the present paper, we consider the non-simply connected medium $\mathcal{M} = \mathbf{R}^3 \backslash \{x = y = 0\}$ with metric (17) as the field of dispiration. In the following, we briefly review some of the geometrical and topological properties that will be useful for later discussion.

Curvature and Frank vector:– The curvature two-form $\mathbf{R}$ is defined in terms of the rotational connection $\mathbf{\Gamma}^{(R)}$,

$$\mathbf{R} = d\mathbf{\Gamma}^{(R)} + \mathbf{\Gamma}^{(R)} \wedge \mathbf{\Gamma}^{(R)}. \tag{19}$$

For the case of dispiration with (13), since $\mathbf{\Gamma}^{(R)} \wedge \mathbf{\Gamma}^{(R)} = 0$, it integrates into

$$\int_S \mathbf{R} = \int_S d\mathbf{\Gamma}^{(R)} = \int_{\partial S} \mathbf{\Gamma}^{(R)} = \int_{\partial S} \frac{\gamma}{2\pi} \mathbf{m}\, d\theta = \gamma \mathbf{m}, \tag{20}$$

where ∂S denotes the boundary of a surface S. Now we choose an orthogonal frame (ξ^1, ξ^2, ξ^3) such that $ds^2 = \delta_{kl} \xi^k \wedge \xi^l$ with $\xi^1 = dr$, $\xi^2 = \sigma r d\theta$, $\xi^3 = dz + \beta d\theta$. Then we have $\mathbf{R} = \mathbf{R}_{kl} \xi^k \wedge \xi^l$. The surface is chosen to be orthogonal to ξ^3. Eq. (20) implies that

$$\mathbf{R} = \gamma\, \mathbf{m}\, \delta^{(2)}(\xi^1, \xi^2)\, d\xi^1 \wedge d\xi^2. \tag{21}$$

The corresponding scalar curvature is

$$R = 2\gamma \delta^{(2)}(\xi^1, \xi^2). \tag{22}$$

or equivalently

$$R = 2\frac{\gamma}{\sigma} \delta^{(2)}(x, y). \tag{23}$$

Apparently the curvature of the medium $\mathcal{M}$ is zero everywhere except along the z-axis ($x = y = 0$). Note that the curvature is created not by the dislocation but by the disclination.

The Frank vector $\mathbf{f} = \{\Phi^{23}, \Phi^{31}, \Phi^{12}\}$ is defined to characterize a disclination with

$$\mathbf{\Phi} = \int_S \mathbf{R} = \gamma\, \mathbf{m} \tag{24}$$

where again the integral is over a surface S which is delimited by a loop ∂S enclosing the z-axis. Obviously the only non-vanishing components of the Frank vector for the dispiration is given by Φ^{12}. As a result, the Frank vector takes the form,

$$\mathbf{f} = \gamma\, \mathbf{e}_z, \tag{25}$$

which, points the z-direction and its magnitude is identical to the deficit angle γ.

Torsion and Burgers vector:– The torsion two-form is defined by

$$\mathbf{T} = d\boldsymbol{\omega} + \mathbf{\Gamma}^{(R)} \wedge \boldsymbol{\omega} = \mathbf{R} \cdot \mathbf{x} + d\mathbf{\Gamma}^{(T)} + \mathbf{\Gamma}^{(R)} \wedge \mathbf{\Gamma}^{(T)}, \tag{26}$$

which, in general, depends not only on the translational connection but also on the rotational connection. For (13) and (15), $\mathbf{\Gamma}^{(R)} \wedge \mathbf{\Gamma}^{(T)} = 0$. Note also that the integration of $\mathbf{R} \cdot \mathbf{x}$ vanishes. Then the torsion two-form integrates into

$$\int_S \mathbf{T} = \int_S d\mathbf{\Gamma}^{(T)} = \int_{\partial S} \mathbf{\Gamma}^{(T)} = b\mathbf{e}_z. \tag{27}$$

from which follows

$$\mathbf{T} = b\, \mathbf{e}_z \delta^{(2)}(\xi^1, \xi^2) d\xi^1 \wedge d\xi^2. \tag{28}$$

We see that the only non-vanishing component of the torsion two-form is in the z-direction and is contributed only by the screw dislocation.

The Burgers vector $\mathbf{b}$ is defined by the surface integral of the torsion two-form

$$\mathbf{b} = \int_S \mathbf{T} \tag{29}$$

which has been evaluated in (27)

$$\mathbf{b} = b\mathbf{e}_z. \tag{30}$$

As is expected the translation parameter b of (6) is indeed the magnitude of the Burgers vector $\mathbf{b}$.

Conical space:– Before closing this section, we consider a special case of the line element (1) with constant z and $\beta = 0$. The two dimensional surface having the metric

$$dl^2 = dr^2 + \sigma^2 r^2 d\theta^2 \tag{31}$$

to which (1) reduces may be realized as a conical surface M_c if the surface is imbedded into a three dimensional Euclidean space E^3. Let

$$\mathbf{X} = \left(\sigma r \cos\theta, \sigma r \sin\theta, \sqrt{1-\sigma^2}\, r\right). \tag{32}$$

Apparently,

$$I: \ d\mathbf{X} \cdot d\mathbf{X} = dl^2 \tag{33}$$

which is the first fundamental form of the imbedded surface M_c. Again, parameterizing the surface by $0 < r < \infty$ and $0 \le \theta < 2\pi$, we have the metric tensor and its inverse,

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 r^2 \end{pmatrix}, \qquad g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^{-2} r^{-2} \end{pmatrix}, \tag{34}$$

which will be used later. With

$$\mathbf{X}_r = \left(\sigma \cos\theta, \sigma \sin\theta, \sqrt{1-\sigma^2}\right), \tag{35}$$

$$\mathbf{X}_\theta = \left(-\sigma r \sin\theta, \sigma r \cos\theta, 0\right), \tag{36}$$

the unit vector normal to M_c at a point (r, θ),

$$\mathbf{n} = \mathbf{X}_r \times \mathbf{X}_\theta / |\mathbf{X}_r \times \mathbf{X}_\theta| \tag{37}$$

is easily calculated to be

$$\mathbf{n} = \left(-\sqrt{1-\sigma^2}\cos\theta, -\sqrt{1-\sigma^2}\sin\theta, \sigma\right). \tag{38}$$

The second fundamental form,

$$II: \ -d\mathbf{X} \cdot d\mathbf{n} = G_{ab} du^a \otimes du^b, \tag{39}$$

is also immediately obtained for the conical surface M_c as

$$-d\mathbf{X} \cdot d\mathbf{n} = \sqrt{1-\sigma^2}\, \sigma r\, d\theta^2. \tag{40}$$

Hence

$$\mathbf{G} = \left(\sum_b g^{ab} G_{bc}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-\sigma^2}/(\sigma r) \end{pmatrix} \tag{41}$$

whose two eigenvalues $k_1 = 0$ and $k_2 = \sqrt{1-\sigma^2}/(\sigma r)$ are the principal curvatures of the conical surface for $r \neq 0$. The Gaussian curvature and the mean curvature of a two dimensional surface imbedded in E^3 are defined, respectively, by

$$K = det\mathbf{G} = k_1 k_2 \tag{42}$$

$$H = \frac{1}{2} tr\mathbf{G} = \frac{1}{2}(k_1 + k_2). \tag{43}$$

For the conical surface in question,

$$K = 0 \qquad \text{and} \qquad H = \sqrt{1-\sigma^2}/(2\sigma r) \tag{44}$$

for $r \neq 0$. Of course the Gaussian curvature K does not vanish at the apex of the cone. According to the Gauss-Bonnet theorem,

$$\int_S K da + \int_{\partial S} k_g dl = 2\pi\chi(S) \tag{45}$$

where S is a compact two-dimensional Riemann manifold with boundary ∂S, K is the Gaussian curvature of S, k_g is the geodesic curvature of ∂S, and $\chi(S)$ is the Euler characteristic of S. Now we let S be the lateral surface of a frustum with slant height r. Then $k_g = 1/r$ and the line element dl integrated along the boundary ∂S with $dr = 0$ result in

$$\int_{\partial S} k_g dl = 2\pi\sigma = 2\pi - \gamma \tag{46}$$

where γ is the deficit angle defined in (18). The Euler characteristic of a cone is unity. Hence we have

$$\int_S K da = \gamma \tag{47}$$

where $da = \sigma r dr\, d\theta$. From this and (44) follows

$$K = \frac{\gamma}{\sigma}\, \delta^2(x, y) \tag{48}$$

which includes the curvature at the apex ($r = 0$). As is well-known, the Ricci or scalar curvature of a two-dimensional manifold is twice the Gaussian curvature. Indeed, twice the Gaussian curvature (48) coincides with the scalar curvature (23).

3 Path integration

The medium surrounding the dispiration considered in the preceding section is $\mathcal{M} = \mathbf{R}^3 \backslash \{x = y = 0\}$, which is geometrically a non-Riemannian space and topologically a non-simply connected space. In this section, we carry out path integration for the propagator of a particle moving in the field of the dispiration. To confine the particle in the vicinity of the dispiration, we introduce a two-dimensional harmonic oscillator potential. Furthermore, we assume a repulsive inverse-square potential to prevent the particle from falling into the singularity at the defect line. For the purpose of comparison, we introduce also a vector potential due to a flux tube and a uniform magnetic field.

The standard approach deals with the Schrödinger equation in curved space. As will be discussed in Sec. 5, the Schrödinger equation is usually modified in curved space not only by the Laplace-Beltrami operator replacing the Laplacian but also the so-called curvature term added as an effective potential. The energy spectrum is sensitive to the type of curvature term; yet the controversy on the choice of the term is not fully settled. The path integral calculation we present suggests that the Gaussian curvature of the surface where the particle moves would dominate the curvature term.

3.1 The Lagrangian

Now that the dispiration field is characterized by the line element (17), the Lagrangian for a charged point particle of mass M moving in the vicinity of the dispiration under the influence of a scalar potential $V(\mathbf{x})$

and a vector potential $\mathbf{A}(\mathbf{x})$ is written as

$$L = \frac{1}{2}M\left(\frac{ds}{dt}\right)^2 - \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) - V(\mathbf{x}). \tag{49}$$

As has been mentioned before, we choose the vector potential $\mathbf{A}(\mathbf{x})$ consisting of two parts; one due to an ideally thin flux tube that contains constant magnetic flux Φ along the dispiration line, and another due to a uniform constant magnetic field $\mathbf{B} = B\,\mathbf{e}_z$ pointing the z-direction. Then the vector potential term of the Lagrangian (49) is expressed in cylindrical coordinates (r, θ, z) as

$$\frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A} = \alpha\hbar\dot{\theta} + M\omega_{_L} r^2\dot{\theta} \tag{50}$$

where

$$\alpha = \frac{e\Phi}{2\pi\hbar c}, \qquad \omega_{_L} = \frac{eB}{2Mc}. \tag{51}$$

Here α is the ratio of the magnetic flux to the fundamental fluxon $\Phi_0 = 2\pi\hbar c/e$, which is identical to the statistical parameter in Wilczek's anyon model [35], and $\omega_{_L}$ is the Lamor frequency. The flux tube is included in our calculation in order to observe the similarity between the Aharonov-Bohm effect and the effect of a screw dislocation. In the scalar potential $V(\mathbf{x})$, we include a two-dimensional short range repulsive potential (the inverse square potential with $\kappa > 0$ sufficiently large) to emphasize the impenetrable feature of the central singularity, and a two-dimensional long range attractive potential (the harmonic oscillator potential) to confine the particle in the vicinity of the dispiration. Namely, $V(\mathbf{x})$ is specified to be a two-dimensional central force potential,

$$V(r) = \frac{\kappa\hbar^2}{8M\sigma^2 r^2} + \frac{1}{2}M\omega_0^2 r^2\ , \qquad r^2 = x^2 + y^2. \tag{52}$$

Thus the Lagrangian we consider is

$$L = \frac{1}{2}M\left\{\dot{r}^2 + \sigma^2 r^2\dot{\theta}^2 + \left(\dot{z} + \beta\dot{\theta}\right)^2\right\} - \alpha\hbar\dot{\theta} - M\omega_{_L} r^2\dot{\theta} - V(r). \tag{53}$$

3.2 The propagator

The transition amplitude (propagator) for the three-dimensional motion of the charged particle from point $\mathbf{x}' = (r', \theta', z')$ to point $\mathbf{x}'' = (r'', \theta'', z'')$ can be calculated by the path integral [21]

$$K(\mathbf{x}'', \mathbf{x}'; \tau) = \int_{\mathbf{x}'=\mathbf{x}(t')}^{\mathbf{x}''=\mathbf{x}(t'')} \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} L\,dt\right]\mathcal{D}^3\mathbf{x} \tag{54}$$

where $\tau = t'' - t' > 0$. The integral measure must be so chosen that the propagator satisfies the properties,

$$\lim_{t''\to t'} K(\mathbf{x}'', \mathbf{x}'; t'' - t') = \delta(\mathbf{x}'' - \mathbf{x}'), \tag{55}$$

$$\int K(\mathbf{x}'', \mathbf{x}; t'' - t)\ K(\mathbf{x}, \mathbf{x}'; t - t')\ d^3\mathbf{x} = K(\mathbf{x}'', \mathbf{x}'; t'' - t')\,. \tag{56}$$

The path integral we calculate with the Lagrangian (53) is

$$\begin{aligned} &K(\mathbf{r}'', z''; \mathbf{r}', z'; \tau) \\ &\quad = \int \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left\{\frac{M}{2}\left(\dot{r}^2 + \sigma^2 r^2\dot{\theta}^2\right) + \frac{M}{2}\left(\dot{z} + \beta\dot{\theta}\right)^2 - \alpha\hbar\dot{\theta} - M\omega_{_L} r^2\dot{\theta} - V(r)\right\}dt\right]\mathcal{D}^2\mathbf{r}\,\mathcal{D}z. \end{aligned} \tag{57}$$

In (57) $\mathbf{r}$ signifies two variables (r, θ) symbolically, and the two dimensional integral measure $d^2\mathbf{r}$ will be specified later. After path integration, from the propagator, we should be able to extract the energy spectrum and the wave functions for the system.

3.3 The z-integration

First we perform the z-integration by letting $\zeta = z + \beta\theta$. The z-path integral is nothing but the Gaussian path integral for a one dimensional free particle, which yields the standard result,

$$\int_{\zeta'=\zeta(t')}^{\zeta''=\zeta(t'')} \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\frac{M}{2}\dot{\zeta}^2 dt\right]\mathcal{D}\zeta = \sqrt{\frac{M}{2\pi i\hbar\tau}}\exp\left[\frac{iM\left(\zeta''-\zeta'\right)^2}{2\hbar\tau}\right]. \tag{58}$$

Now we rewrite the right hand side of (58) as

$$\sqrt{\frac{M}{2\pi i\hbar\tau}}\exp\left[\frac{iM\left(\zeta''-\zeta'\right)^2}{2\hbar\tau}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tau\hbar k^2/2M} e^{i(\zeta''-\zeta')k} dk \tag{59}$$

where $\hbar k$ is the z-component of momentum of the particle. We also notice that

$$\zeta''-\zeta' = z''-z'+\beta'\left(\theta''-\theta'\right) = z''-z'+\beta\int_{t'}^{t''}\dot{\theta}dt. \tag{60}$$

Incorporating these results into the path integral (57), we decompose the propagator as

$$K\left(\mathbf{x}'',\mathbf{x}';\tau\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik\left(z''-z'\right)} e^{-i\tau\hbar k^2/2M} K^{(k)}\left(\mathbf{r}'',\mathbf{r}';\tau\right) \tag{61}$$

with the two-dimensional propagator for a fixed k value,

$$\begin{aligned} &K^{(k)}\left(\mathbf{r}'',\mathbf{r}';\tau\right) \\ &= \int_{\mathbf{r}'=\mathbf{r}(t')}^{\mathbf{r}''=\mathbf{r}(t'')} \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\left(\dot{r}^2+\sigma^2 r^2\dot{\theta}^2\right) - M\omega_{_L} r^2\dot{\theta} - \xi\hbar\dot{\theta} - V(r)\right]dt\right\}\mathcal{D}^2\mathbf{r}. \end{aligned} \tag{62}$$

where $\xi = \alpha - \beta k$. The integral on the right hand side of (59) is the spectral decomposition of the z-motion with the continuous spectrum,

$$E_k = \frac{\hbar^2 k^2}{2M}. \tag{63}$$

Next we make a change of the angular variable from θ to ϑ by letting

$$\dot{\vartheta} = \dot{\theta} - \bar{\omega} \tag{64}$$

where $\bar{\omega} = \omega_{_L}/\sigma^2$. Accordingly the k-propagator (62) is transformed into

$$\begin{aligned} &K^{(k)}\left(\mathbf{r}'',\mathbf{r}';\tau\right) \\ &= \int_{\mathbf{r}'=\mathbf{r}(t')}^{\mathbf{r}''=\mathbf{r}(t'')} \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\left(\dot{r}^2+\sigma^2 r^2\dot{\vartheta}^2\right) - \xi\hbar\dot{\vartheta} - U(r)\right]dt\right\}\mathcal{D}^2\mathbf{r}(r,\vartheta) \end{aligned} \tag{65}$$

where

$$U(r) = V(r) + \frac{1}{2}M\bar{\omega}^2 r^2 - \xi\hbar\bar{\omega}, \tag{66}$$

or

$$U(r) = \frac{\kappa\hbar^2}{8M\sigma^2 r^2} + \frac{1}{2}M\omega^2 r^2 + V_0 \tag{67}$$

where

$$\omega^2 = \omega_0^2 + \bar{\omega}^2, \qquad V_0 = -\xi\hbar\bar{\omega}. \tag{68}$$

Note that ϑ is the angular variable in a rotating frame with angular velocity $-\bar{\omega}$. For simplicity the constant V_0 in the potential will be ignored in the calculation below.

3.4 The short time action

To calculate the two-dimensional path integral (65) in polar coordinates [22, 23, 24], we first express it in discretized form,

$$K^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau) = \lim_{N\to\infty} \int_{\mathbf{r}'=\mathbf{r}(t')}^{\mathbf{r}''=\mathbf{r}(t'')} \prod_{j=1}^{N} K^{(k)}(\mathbf{r}_j, \mathbf{r}_{j-1}; \epsilon) \prod_{j=1}^{N-1} d^2\mathbf{r}_j \tag{69}$$

where the propagator for a short time interval $\epsilon = t_j - t_{j-1} = \tau/N$ is given by

$$K^{(k)}(\mathbf{r}_j, \mathbf{r}_{j-1}; \epsilon) = A_j \exp\left(\frac{i}{\hbar} S_j\right). \tag{70}$$

Now we select a relevant approximation of the short time action S_j in cylindrical coordinates, and determine the amplitude A_j so as to meet the normalization condition

$$\lim_{\epsilon\to 0} K^{(k)}(\mathbf{r}_j, \mathbf{r}_{j-1}; \epsilon) = \delta^{(2)}(\mathbf{r}_j - \mathbf{r}_{j-1}) \tag{71}$$

where the two-dimensional delta function satisfies

$$\int \delta^{(2)}(\mathbf{r}_j - \mathbf{r}_{j-1})\, d^2\mathbf{r}_j = 1. \tag{72}$$

The short time action is

$$S_j = \int_{t_{j-1}}^{t_j} \left[\frac{M}{2}(\dot{r}^2 + \sigma^2 r^2 \dot{\vartheta}^2) - \xi\hbar\dot{\vartheta} - U(r)\right] dt \tag{73}$$

which we approximate by

$$S_j = \frac{M}{2\epsilon}\left\{(\Delta r_j)^2 + 2\sigma^2 r_j r_{j-1}\left[1 - \cos(\Delta\vartheta_j)\right]\right\} - \xi\hbar\Delta\vartheta_j - U_j\epsilon\ , \tag{74}$$

where $\Delta r_j = r_j - r_{j-1}$, $\Delta\vartheta_j = \vartheta_j - \vartheta_{j-1}$, and $U_j = U(r_j)$. It is tempting to approximate $(d\vartheta)^2$ by $(\Delta\vartheta)^2$. In path integration, however, $(\Delta\vartheta)^4/\epsilon$ cannot be ignored because $(\Delta\vartheta)^2 \sim \epsilon$. Hence it is appropriate to replace $(d\vartheta)^2$ by $1 - \cos(\Delta\vartheta)$ even though unimportant higher order terms are included (see [25]). Since it is sufficient for a short time action to consider the contributions up to first order in ϵ, we further employ the approximate relation for small ϵ [15],

$$\cos(\Delta\vartheta) + a\epsilon\Delta\vartheta \sim \cos(\Delta\vartheta - a\epsilon) + \frac{1}{2}a^2\epsilon^2 \tag{75}$$

to write the short time action multiplied by $(i/\hbar)$ as

$$\begin{aligned}\frac{i}{\hbar}S_j \quad &= \quad \frac{iM}{2\hbar\epsilon}\left(r_j^2 + r_{j-1}^2\right) + (\sigma^{-2} - 1)\frac{M\sigma^2 r_j r_{j-1}}{i\hbar\epsilon} \\ &\quad + \frac{M\sigma^2 r_j r_{j-1}}{i\hbar\epsilon}\cos\left(\Delta\vartheta_j - \frac{\xi\hbar\epsilon}{M\sigma^2 r_j r_{j-1}}\right) + \frac{(4\xi^2 + \kappa)\hbar\epsilon}{8Mi\sigma^2 r_j r_{j-1}} - \frac{iM\omega^2\epsilon}{4\hbar}(r_j^2 + r_{j-1}^2).\end{aligned} \tag{76}$$

We use this short time action (multiplied by $i/\hbar$) for evaluating the angular path integral. However, before starting the angular integration, let us determine the amplitude A_j by considering the limit $\epsilon \to 0$ of the short time action (76),

$$\begin{aligned}\lim_{\epsilon\to 0} A_j e^{iS_j/\hbar} \quad &= \quad \lim_{\epsilon\to 0} A_j \exp\left\{\frac{iM}{2\hbar\epsilon}(\Delta r_j)^2\right\} \exp\left\{\frac{iM\sigma^2 r_j^2}{2\hbar\epsilon}(\Delta\vartheta_j)^2\right\} \\ &= \quad A_j \frac{2\pi i\hbar\epsilon}{M\sigma r_j}\delta(\Delta r_j)\,\delta(\Delta\vartheta_j).\end{aligned} \tag{77}$$

Now let the areal element be given by

$$d^2\mathbf{r}_j = r_j\, dr_j\, d\vartheta_j. \tag{78}$$

Then the amplitude meeting the condition (71) must be of the form,

$$A_j = \frac{M\sigma}{2\pi i\hbar\epsilon}. \tag{79}$$

Alternatively, if

$$d^2\mathbf{r}_j = \sigma r_j\, dr_j\, d\vartheta_j, \tag{80}$$

then the corresponding amplitude is

$$A_j = \frac{M}{2\pi i\hbar\epsilon}. \tag{81}$$

Since path integration with either combination yields the same result, we employ the first choice (78) with (79) for our calculation.

3.5 Asymptotic recombination

For path integration in polar coordinates, separation of variables is not straightforward. To separate the angular variable from the radial function, we employ the asymptotic recombination technique (see [25]). The asymptotic form of the modified Bessel function $I_\nu(z)$ for large $|z|$ (GR:8.451.5 in [26]) is

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} (-1)^n \frac{(\nu, n)}{(2z)^n} + \frac{e^{-z-(\nu+(1/2))\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2z)^n} \tag{82}$$

where $-3\pi/2 < \arg z < \pi/2$, and

$$(\nu, n) = \frac{\Gamma(\nu + n + \frac{1}{2})}{n!\,\Gamma(\nu - n + \frac{1}{2})}, \qquad (\nu, 0) = 1.$$

The asymptotic recombination technique is based on the conjecture that the one-term asymptotic form,

$$I_\nu\,(z) \sim \frac{1}{\sqrt{2\pi z}} \exp\left[z - \frac{1}{2z}\left(\nu^2 - \frac{1}{4}\right)\right], \tag{83}$$

is valid for sufficiently large $|z|$ and for $-\pi/2 < \arg z < \pi/2$ as relevant in path integration [22, 27]. With the help of the one-term form which we refer to as the Edwards-Gulyaev asymptotic formula, we can derive the following asymptotic relation for large $|z|$ [25],

$$I_\nu(az)\, e^{bz}\, e^{-c/z} \sim \sqrt{\frac{a+b}{a}}\, I_\mu\left[(a+b)z\right], \tag{84}$$

where $a > 0$, $b \geq 0$, $8a(a+b)c > b - 4(a+b)\nu^2$ and

$$\mu = \left[\frac{a+b}{a}\nu^2 - \frac{b}{4a} + 2(a+b)c\right]^{1/2}. \tag{85}$$

Making use of this asymptotic relation together with the Jacobi-Anger expansion formula

$$e^{z\cos\vartheta} = \sum_{m=-\infty}^{\infty} e^{im\vartheta} I_m(z)\ , \tag{86}$$

we obtain another asymptotic relation for large z,

$$\exp\left\{bz + z\cos\left[\Delta\vartheta + i\frac{d}{z}\right] - \frac{(d^2+2f)}{2z}\right\} \sim \sqrt{1+b}\sum_{m=-\infty}^{\infty} e^{im\,\Delta\vartheta}\, I_\mu\left[(1+b)z\right] \tag{87}$$

with

$$\mu = \left[(1+b)\{(m+d)^2+2f\} - b/4\right]^{1/2}. \tag{88}$$

In the above we have let $a=1$ and $c=(d^2+2f+2md)/2$.

3.6 Angular integration

Utilizing (87) with $b=\sigma^{-2}-1$, $z=M\sigma^2 r_j r_{j-1}/(i\hbar\epsilon)$, $d=\xi=\alpha-\beta k$ and $f=\kappa/2$, we separate variables of the short time propagator (70) as

$$K^{(k)}\left(\mathbf{r}_j,\mathbf{r}_{j-1};\epsilon\right) = A_j\exp\left(\frac{i}{\hbar}S_j\right) = \frac{1}{2\pi}\sum_{m_j=-\infty}^{\infty} e^{im_j(\vartheta_j-\vartheta_{j-1})}R_{m_j}\left(r_j,r_{j-1};\epsilon\right) \tag{89}$$

with the short time radial propagator,

$$R_{m_j}\left(r_j,r_{j-1};\epsilon\right) = \frac{M}{i\hbar\epsilon}\exp\left[\frac{iM}{2\hbar\epsilon}\left(r_j^2+r_{j-1}^2\right) - \frac{iM\omega^2\epsilon}{4\hbar}(r_j^2+r_{j-1}^2)\right] I_{\mu(m_j)}\left(\frac{Mr_jr_{j-1}}{i\hbar\epsilon}\right) \tag{90}$$

where

$$\mu(m_j) = \frac{1}{2\sigma}\left[4(m_j+\xi)^2+\sigma^2-1+\kappa\right]^{1/2}. \tag{91}$$

The prefactor $\sqrt{1+b}$ appeared on the right hand side of the formula (87) results in a multiple factor σ^{-1} before the summation of (89), which will cancel out the factor σ appearing in the amplitude (79). Thus the two-dimensional path integral (69) is reduced to the form,

$$K^{(k)}\left(\mathbf{r}'',\mathbf{r}';\tau\right) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\left[\frac{1}{2\pi}\sum_{m_j} e^{im_j(\vartheta_j-\vartheta_{j-1})}\,R_{m_j}(r_j,r_{j-1};\epsilon)\right]\prod_{j=1}^{N-1} r_j dr_j\, d\vartheta_j. \tag{92}$$

The angular integration can be done straightforwardly by using the orthogonality relation,

$$\int_0^{2\pi} e^{i(m'-m)\vartheta}\,d\vartheta = 2\pi\delta_{m',m}. \tag{93}$$

Namely,

$$\int\prod_{j=1} e^{im_j\Delta\vartheta_j}\prod_{j=1}^{N-1} d\vartheta_j = (2\pi)^{N-1}\prod_{j=1}^{N-1}\delta_{m,m_j}\,e^{im(\vartheta''-\vartheta')}. \tag{94}$$

where $m=m_N$. After angular integration, we have $m_j=m$ for all j, and arrive at the expression for the full propagator with a fixed wave number k,

$$K^{(k)}\left(\mathbf{r}'',\mathbf{r}';\tau\right) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp\left[im\left(\vartheta''-\vartheta'\right)\right]R_m(r'',r';\tau) \tag{95}$$

where

$$R_m(r'',r';\tau) = \lim_{N\to\infty}\int\prod_{j=1}^{N} R_m\left(r_j,r_{j-1};\epsilon\right)\prod_{j=1}^{N-1} r_j dr_j. \tag{96}$$

The left hand side of (95) is nothing but the partial wave expansion in two dimensions, and the radial propagator (96) corresponds to the m-th partial wave propagator. However, the last radial path integration (96) remains to be carried out.

3.7 Radial path integration

To perform the radial path integration explicitly, we first notice that the short time radial propagator (90) with $m = m_j$ for all j is similar in form as that of the radial harmonic oscillator. Then we rewrite (96) as

$$R_m(r_j, r_{j-1}; \epsilon) = \frac{M}{i\hbar\epsilon} \exp\left[\frac{iM\omega}{2\hbar}(r_j^2 + r_{j-1}^2)\frac{1}{\omega\epsilon}\left(1 - \frac{1}{2}\omega^2\epsilon^2\right)\right] I_{\mu(m)}\left(\frac{Mr_jr_{j-1}}{i\hbar\epsilon}\right). \tag{97}$$

We further simplify (97) by letting $\eta_j = (M\omega/2\hbar)r_j^2$ and $\varphi = \arcsin(\omega\epsilon)$ as

$$R_m(r_j, r_{j-1}; \epsilon) = \frac{M\omega}{i\hbar\sin\varphi} \exp\left[i(\eta_j + \eta_{j-1})\cot\varphi\right] I_\mu\left(-2i\sqrt{\eta_j\eta_{j-1}}\,\csc\varphi\right). \tag{98}$$

Here we have used the approximation,

$$\cos\varphi_j = \cos[\arcsin(\omega\epsilon)] \sim 1 - \frac{1}{2}\omega^2\epsilon^2. \tag{99}$$

At this point, for convenience, we introduce a two point function, referred to as the υ-function [25, 28], by

$$\upsilon_\mu\left(\eta, \eta'; \varphi\right) = -i\csc\varphi \exp\left[i\left(\eta + \eta'\right)\cot\varphi\right] I_\mu\left(-2i\sqrt{\eta\eta'}\csc\varphi\right) \tag{100}$$

satisfying the convolution relation

$$\int_0^\infty \upsilon_\mu\left(\eta'', \eta; \varphi\right)\upsilon_\mu\left(\eta, \eta'; \varphi\right) d\eta = \upsilon_\mu\left(\eta'', \eta'; 2\varphi\right) \tag{101}$$

which can be derived from Weber's formula (GR: 6.633.2 in [26]) as modified in [23],

$$\int_0^\infty \exp(i\alpha r^2)\, I_\mu(-iar)\, I_\mu(-ibr)\, r\, dr = \frac{i}{2\alpha}\exp\left[-\frac{i}{4\alpha}(a^2 + b^2)\right] I_\mu\left(-\frac{ab}{2\alpha}\right) \tag{102}$$

valid for Re $\alpha > 0$ and Re $\mu > -1$. Then the short time radial function can be expressed in terms of the υ-function as

$$R_m(r_j, r_{j-1}; \epsilon) = \frac{M\omega}{\hbar}\upsilon_{\mu(m)}\left(\eta_j, \eta_{j-1}; \varphi\right). \tag{103}$$

Now we are ready to perform the radial path integration. Substitution of this into (96) gives

$$R_m\left(r'', r'; \tau\right) = \frac{M\omega}{\hbar}\lim_{N\to\infty}\int\prod_{j=1}^{N}\upsilon_{\mu(m)}\left(\eta_j, \eta_{j-1}; \varphi\right)\prod_{j=1}^{N-1}d\eta_j. \tag{104}$$

The convolution property (101) of the υ-function enables us to reach

$$R_m\left(r'', r'; \tau\right) = \frac{M\omega}{\hbar}\upsilon_{\mu(m)}\left(\eta'', \eta'; \omega\tau\right). \tag{105}$$

In the above we have also used the property,

$$\lim_{N\to\infty}(N\varphi) = \lim_{N\to\infty}(N\,\sin\varphi) = \lim_{N\to\infty}(\omega N\epsilon) = \omega\tau. \tag{106}$$

In terms of the modified Bessel function it is written as

$$R_m\left(r'', r'; \tau\right) = \frac{M\omega}{i\hbar\sin\omega\tau}\exp\left[\frac{iM\omega}{2\hbar}\left(r'^2 + r''^2\right)\cot\omega\tau\right] I_{\mu(m)}\left(\frac{M\omega r'r''}{i\hbar\sin\omega\tau}\right) \tag{107}$$

with

$$\mu(m) = \frac{1}{2\sigma}\sqrt{4(m+\alpha-\beta k)^2+\sigma^2-1+\kappa}. \tag{108}$$

Notice that $\mu(m)$ is a real positive number if $1-\sigma^2 < \kappa$. In this manner we have completed the radial path integration for the partial propagator with k fixed. It turns out that the radial propagator we have obtained above is identical in form with that of the radial harmonic oscillator. The characteristics of the dispiration, the flux tube, the uniform magnetic field and the assumed potential, which make the present system different from the simple harmonic oscillator, are all taken into the index $\mu(m)$ of the modified Bessel function.

With the radial propagator (107), the full propagator with k fixed is obtained by

$$\begin{aligned} &K^{(k)}\left(r'',\vartheta'';r',\vartheta';\tau\right) \\ &\quad= \frac{M\omega}{2\pi i\hbar\sin\omega\tau}\exp\left[\frac{iM\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega\tau\right]\sum_{m=-\infty}^{\infty} e^{im(\vartheta''-\vartheta')}\, I_{\mu(m)}\left(\frac{M\omega r'r''}{i\hbar\sin\omega\tau}\right) \end{aligned} \tag{109}$$

or by converting ϑ into $\theta = \vartheta + \bar{\omega}t$,

$$\begin{aligned} &K^{(k)}\left(r'',\theta'';r',\theta';\tau\right) \\ &\quad= \frac{M\omega}{2\pi i\hbar\sin\omega\tau}\exp\left[\frac{iM\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega\tau\right]\sum_{m=-\infty}^{\infty} e^{im(\theta''-\theta'-\bar{\omega}\tau)}\, I_{\mu(m)}\left(\frac{M\omega r'r''}{i\hbar\sin\omega\tau}\right). \end{aligned} \tag{110}$$

4 Winding number expansion

In the angular path integration performed above, we have not explicitly taken account of the topological structure of the background medium $\mathcal{M}$. Since $\mathcal{M} = \mathbf{R}^3\backslash\{x=y=0\} \cong \mathbf{R}\times\mathbf{R}^+\times S^1$, the paths connecting two points, say a and b, in $\mathcal{M}$ can wind around the z-axis many times, and may be classified (into homotopy classes) by the fundamental group $\pi_1(\mathcal{M}) = \pi_1(S^1) \cong \mathbf{Z}$ of $\mathcal{M}$. A set of all homotopically equivalent paths in $\mathcal{M}$ is now characterized by a single winding number $n \in \mathbf{Z}$. Therefore the propagator $K(b,a)$ may be given as a sum of subpropagators $\tilde{K}_n(b,a)$ for the paths with different winding numbers n,

$$K(b,a) = \sum_{n\in\mathbf{Z}} C_n\, \tilde{K}_n(b,a). \tag{111}$$

The paths in a same class may be deformed into one another, so that the transition amplitudes corresponding to these paths must share the same phase factor, On the other hand, the paths belonging to different classes may have different phases. Laidlaw and deWitt [29], and Schulman [13] argue that the coefficients C_n are the one-dimensional unitary representations of the fundamental group $\pi_1(\mathcal{M}) \cong \mathbf{Z}$. From eq. (111) it is apparent that $C_{n+m} = C_m C_n$. If no degeneracies and no internal degrees of freedom are assumed, $C_{n+1} = e^{i\delta}C_n$. Then the coefficients are given by the one-dimensional representations of the fundamental group $\pi_1(\mathcal{M})$ or the additive group $\mathbf{Z}$, namely $C_n = e^{in\delta}$.

It has also been pointed out [30] that the angular momentum representation and the winding number representation are complementary to each other via Poisson's sum formula,

$$\sum_{n\in\mathbf{Z}} e^{2\pi in\xi} = \sum_{m\in\mathbf{Z}} \delta(\xi - m). \tag{112}$$

This means that the propagator $K^{(k)}(\mathbf{r}'',\mathbf{r}';\tau)$ we have obtained in the preceding sections, as is given in the angular momentum representation, cannot be regarded as a subpropagator carrying a single winding number. Similarly, any partial propagator with a fixed angular momentum quantum number cannot be decomposed to a full set of subpropagators with winding numbers. The propagator obtained for the dispiration field must

be understood as a full bound state propagator that contains contributions from all homotopically possible paths (see [15, 16] for detail). Thus we look for a winding number representation of $K^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau)$ in (109) via Poisson's formula (112).

Let us rewrite (95) as

$$K^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau) = \frac{1}{2\pi} \int \sum_{m \in \mathbf{Z}} \delta(\alpha' + \lambda - m)\, e^{i(\alpha'+\lambda)(\vartheta''-\vartheta')}\, R_{\alpha'+\lambda}(r'', r'; \tau)\, d\lambda \tag{113}$$

with the radial propagator $R_m(r'', r'; \tau)$ given by (107). Then we utilize Poisson's formula (112) to convert (113) into the winding number representation

$$K^{(k)}(\mathbf{r}'', \mathbf{r}'; \tau) = \sum_{n \in \mathbf{Z}} C_n \tilde{K}_n(\mathbf{r}'', \mathbf{r}'; \tau). \tag{114}$$

where

$$C_n = e^{i2\pi n \alpha'} \tag{115}$$

and

$$\begin{aligned} &\tilde{K}_n(\mathbf{r}'', \mathbf{r}'; \tau) \\ &\quad = \frac{M\omega e^{i\alpha'(\vartheta''-\vartheta')}}{2\pi i\hbar \sin\omega\tau} \exp\left[\frac{iM\omega}{2\hbar}\left(r'^2 + r''^2\right)\cot\omega\tau\right] \int_{-\infty}^{\infty} e^{i\lambda(\vartheta''-\vartheta'-2\pi n)}\, I_{\mu(\alpha'+\lambda)}\left(\frac{M\omega r' r''}{i\hbar \sin\omega\tau}\right) d\lambda. \end{aligned} \tag{116}$$

In the coefficients C_n of (115) α' can chosen to be an arbitrary real number. In particular, choosing $\alpha' = -\alpha$ one observes that the magnetic flux only appears as a pure phase factor $\exp\{-i\alpha(\vartheta'' - \vartheta' + 2\pi n)\}$ in (116). The choice $\alpha' = \beta k - \alpha$ furthermore shows that the screw dislocation has a similar effect, that is, it only appears as a phase $\exp\{-i(\alpha - k\beta)(\vartheta'' - \vartheta' + 2\pi n)\}$ in (116). Evidently both belong to $U(1)$, the one-dimensional unitary representation of $\pi_1(\mathcal{M})$. The result is consistent with the Laidlaw-deWitt-Schulman theorem albeit the unitary factor is not unique.

5 Energy spectrum and wave functions

All the information concerning the energy spectrum and associated radial wave functions for the bound states in two dimensions is contained in the radial propagator (107). In order to extract such information out of (107), we make use of the Hille-Hardy formula (GR: 8.976.1 in [26]),

$$I_\mu\left(\frac{2\sqrt{xyz}}{1-z}\right)\frac{(xyz)^{-\mu/2}}{1-z}\exp\left[-\frac{1}{2}(x+y)\frac{1+z}{1-z}\right] = \exp\left[-\frac{x+y}{2}\right]\sum_{n=0}^{\infty}\frac{n!\, z^n}{\Gamma(n+\mu+1)}\, L_n^{(\mu)}(x)\, L_n^{(\mu)}(y) \tag{117}$$

where $L_n^{(\mu)}(x)$ is the Laguerre polynomial related to the confluent hypergeometric function as

$$L_n^{(\mu)}(x) = \frac{\Gamma(n+\mu+1)}{\Gamma(\mu+1)\, n!} F(-n, \mu+1; x). \tag{118}$$

Notice that the confluent hypergeometric function is in general an infinite series,

$$F(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)}{\Gamma(b+n)\, s!}\, x^s \tag{119}$$

which converges only for $|x| < 1$ and becomes a polynomial for any x when $a = 0, -1, -2, \ldots$. Letting $x = (M\omega/\hbar)r'^{\,2}$, $y = (M\omega/\hbar)r''^{\,2}$, and $z = e^{-2i\omega\tau}$ in (117), we write (107) as

$$\begin{aligned} R_m\left(r'', r'; \tau\right) &= \frac{2M\omega}{\hbar}\left(\frac{M\omega}{\hbar}r'r''\right)^{\mu} \exp\left[-\frac{M\omega}{2\hbar}\left(r'^{\,2}+r''^{\,2}\right)\right] \\ &\quad \times \sum_{n=0}^{\infty} \frac{n!\, e^{-i\tau\omega(2n+\mu+1)}}{\Gamma(n+\mu+1)}\, L_n^{(\mu)}\left(\frac{M\omega}{\hbar}r'^{\,2}\right) L_n^{(\mu)}\left(\frac{M\omega}{\hbar}r''^{\,2}\right). \end{aligned} \tag{120}$$

With this radial propagator the k-propagator (110) can be cast into the form

$$K^{(k)}(r'', \theta''; r', \theta'; \tau) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \psi_{mn}(r'', \theta'')\psi^*_{mn}(r', \theta')\, e^{-i\tau\tilde{E}_{mn}/\hbar}, \tag{121}$$

where

$$\tilde{E}_{mn} = \hbar\omega(2n + \mu(m) + 1) + m\hbar\bar{\omega} \tag{122}$$

and

$$\psi_{mn}(r, \theta) = \sqrt{\frac{M\omega}{\pi\hbar}}\sqrt{\frac{n!}{\Gamma(n+\mu+1)}}\left(\frac{M\omega}{\hbar}r^2\right)^{\mu/2} e^{-(M\omega/2\hbar)r^2}\, L_n^{(\mu)}\left(\frac{M\omega}{\hbar}\, r^2\right) e^{im\theta}. \tag{123}$$

Substitution of (91) into (122) results in the energy spectrum for the particle bound in two dimensions

$$\tilde{E}_{mn} = \hbar\omega\left\{2n + 1 + \frac{1}{2\sigma}\sqrt{4(m+\alpha-\beta k)^2 + \sigma^2 - 1 + \kappa}\right\} + m\hbar\bar{\omega} \tag{124}$$

where $n = 0, 1, 2, \ldots$, $m = 0, \pm 1, \pm 2, \ldots$. Adding the continuous spectrum (63) as well as the ignored V_0 yields the full spectrum of the system,

$$E_{mnk} = \tilde{E}_{mn} + \frac{\hbar^2 k^2}{2M} + (\beta k - \alpha)\hbar\bar{\omega}. \tag{125}$$

6 Concluding Remarks

In concluding the present paper, we would like to make some remarks on the discrete energy spectrum (124) for the two-dimensional motion around the dispiration. First we examine special cases.

(i) *The Landau levels*: The presence of the uniform magnetic field is unimportant for the study on the dispiration. However, if there are no dislocation, no disclination, no magnetic tube, no external short-ranged repulsive and long-ranged attractive potential but the uniform magnetic field, that is, if $\alpha = \beta = \kappa = 0$, $\sigma = 1$ and $\omega_0 = 0$, then, as is expected, we have the Landau levels,

$$E_{\bar{n},k} = 2\hbar\omega_{_L}\left(\bar{n} + \frac{1}{2}\right) + \frac{\hbar^2 k^2}{2M} \tag{126}$$

where $\omega_{_L} = eB/(2M)$ and $\bar{n} = n + (|m| + m)/2 = 0, 1, 2, \cdots$.

(ii) *The screw dislocation spectrum*: In the absence of the uniform magnetic field, the disclination and the inverse-square potential, i.e., in the case of $\bar{\omega} = 0$, $\sigma = 1$ (i.e. $\kappa = 0$), the discrete energy spectrum for the two-dimensional motion (with fixed k) becomes

$$\tilde{E}_{mn} = \hbar\omega_0\left(2n + 1 + |m + \alpha - \beta k|\right), \tag{127}$$

which shows that the effect of the Burgers vector $b = 2\pi\beta$ is practically identical to that of the magnetic tube (the Aharonov-Bohm effect) as pointed out in ref. [11]. In comparison with Wilczek's anyon model [33],

the screw dislocation plays a similar role of an anyon by generating a fractional spin. From the geometrical point of view, as we have seen earlier, the line dislocation causes torsion. This spectrum explicitly shows that a source of torsion generates a spin effect [31, 32]. In the limiting case of the vanishing flux tube and the diminishing Burgers vector, we have simply the harmonic oscillator spectrum,

$$E_{\bar{n}} = \hbar\omega_0(\bar{n} + 1), \tag{128}$$

where $\bar{n} = 2n + |m| = 0, 1, 2, \cdots$. The harmonic oscillator potential is important in the present dislocation spectrum, without which two-dimensional bound states cannot be formed in the vicinity of the defect. In the calculation of the partition function for an anyon gas, the oscillator potential has played a role of the regulator for taming divergences [33, 34].

(iii) *The wedge disclination spectrum*: If $\alpha = 0$, $\beta = 0$, and $\bar{\omega} = 0$, then the two-dimensional discrete spectrum takes the form,

$$\tilde{E}_{mn} = \hbar\omega_0 \left\{ 2n + 1 + \frac{1}{2\sigma}\sqrt{4m^2 + \sigma^2 - 1 + \kappa} \right\}, \tag{129}$$

which belongs to a particle bound near the disclination by the harmonic oscillator potential plus a repulsive inverse-square potential. From (18) it is clear that $\sigma = 1$ implies the vanishing deficit angle $\gamma = 0$. This corresponds to the absence of disclination. If $0 < \sigma < 1$, then $2\pi > \gamma > 0$, that is, the medium carries a positive curvature at the center of disclination. In comparison with the assumed square inverse repulsive potential term with κ inside the square root of the spectrum, we see that the negative term with $\sigma^2 - 1$ represents the effect of an attractive force; we may argue that the disclination effectively generates a short-ranged attractive force around it. If $\sigma > 1$, the medium is negatively curved, and a repulsive force is created around the center of disclination (the saddle point).

The Schrödinger equations for the harmonic oscillator interacting separately with dislocation and disclination have been solved by Furtado and Moraes [36]. In the absence of the flux tube $\alpha = 0$, our dislocation spectrum (127) coincides with their result obtain for the dislocation. However, our disclination spectrum (129) with $\kappa = 0$ is not in agreement with theirs.

Finally, we examine for the case of disclination the difference between the result from the Schrödinger equation and that of path integration. Since some errors are involved in [36], we present our own solution of the Schrödinger equation which is slightly different from that in [36]. The line element (1), if $\beta = 0$, reads

$$ds^2 = dr^2 + \sigma^2\, r^2\, d\theta^2 + dz^2. \tag{130}$$

Although the Laplace-Beltrami operator can be used to write down the Schrödinger equation as in [36], we choose to let $\phi = \sigma\theta$, and express (130) as

$$ds^2 = dr^2 + r^2\, d\phi^2 + dz^2, \tag{131}$$

which is identical with the flat space line element except for $0 \le \phi < 2\pi\sigma$. Then it is rather trivial to write down the Schrödinger equation in terms of coordinates (r, ϕ, z); namely

$$-\frac{\hbar^2}{2M}\left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right\}\psi + V(r)\psi = E\psi \tag{132}$$

which can be easily solved for a two-dimensional harmonic oscillator potential $V(r) = \frac{1}{2}M\omega^2 r^2$ with the periodic condition $\psi(r, 2\pi\sigma + \phi, z) = \psi(r, \phi, z)$. The normalizable solution is obtained in the form,

$$\psi(r, \phi, z) = N\, e^{ikz} e^{im\phi/\sigma} e^{-M\omega r^2/2\hbar}\, r^{|m|/\sigma}\, F(-n_r, 1 + |m|/\sigma; (M\omega/\hbar) r^2), \tag{133}$$

with the condition

$$\frac{1}{2}\left(1 + \frac{|m|}{\sigma} - \frac{E}{\hbar\omega} + \frac{k^2\hbar^2}{2M} \right) = -n_r \qquad (n_r \in \mathbf{N}_0), \tag{134}$$

which yields the energy spectrum,

$$E_{n_r mk} = \hbar\omega \left\{ 2n_r + 1 + \frac{|m|}{\sigma} \right\} + \frac{k^2\hbar^2}{2M}. \tag{135}$$

Here $m \in \mathbf{Z}$ as follows from the periodic condition. It is apparent that the above spectrum differs from the path integral result (129). The term $\sigma^2 - 1$ inside the square root of (129) lacks in (135). Furthermore, the wave functions are different. The wave functions (133) with $m \neq 0$ vanish at $r = 0$, but the function with $m = 0$ remains to be non-zero. This is in contrast with the fact that the radial wave function (123) obtained by path integration vanishes at $r = 0$ for all values of m. In this treatment the nonvanishing singular curvature at the disclination center $r = 0$ plays no role. The Schrödinger equation may have to be modified so as to accommodate the curvature effect.

In general, the Schrödinger equation in curved space is written in the form,

$$\left\{ -\frac{\hbar^2}{2M}\Delta + V_c(\mathbf{r}) + V(\mathbf{r}) \right\} \psi(\mathbf{r}, t) = i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r}, t) \tag{136}$$

where Δ is the Laplace-Beltrami operator, $V_c(\mathbf{r})$ is the potential due to the curvature effect and $V(\mathbf{r})$ is any external potential. Historically, Podolsky [37] defined the Schrödinger equation in curved space without the curvature term, i.e., $V_c(\mathbf{r}) = 0$, but, as Schulman [14] puts it, there is no reason, other than the prejudice for simplicity, to ignore the curvature term. Comparing with Feynman's path integral, DeWitt [38] has proposed that the scalar curvature term is needed, that is, $V_c(\mathbf{r}) = g\hbar^2 R(\mathbf{r})$ where g is a constant. More recently, however, Kleinert [39] has argued by using a quantum equivalence principle that there is no need of the curvature term. On the other hand, viewing that the motion in a two-dimensional curved space as a constrained motion on the curved surface imbedded in a three-dimensional Euclidean space, Jensen and Koppe [40], da Costa [41] and others have argued that the Schrödinger equation on a curved surface carries in it an effect potential due to the Gaussian curvature $K(\mathbf{r})$ and the mean curvature $H(\mathbf{r})$.

Our path integral calculation necessitates an effective potential of the inverse square form which is due to neither the scalar curvature nor the Gaussian curvature. In a forth coming paper [42] it will be shown that the path integration in a conical space with $K = 0$ and $H = \sqrt{1-\sigma^2}/(2\sigma r)$ for $r \neq 0$ is compatible with the Schrödinger equation modified with the mean curvature H of the conical surface as suggested in [40].

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Figures

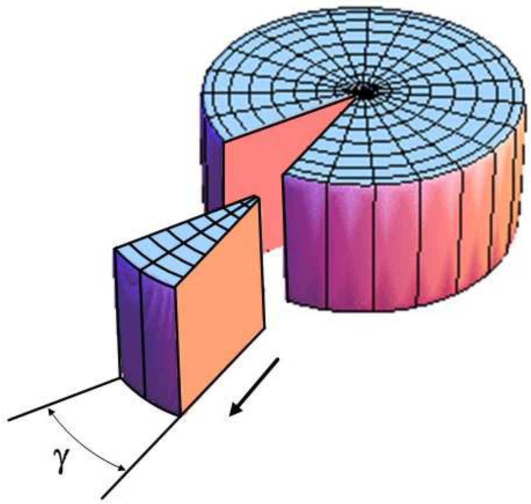

Figure 1: Make a thin cylindrical tube along the central line and remove a wedge of angle $\gamma > 0$ from the main body.

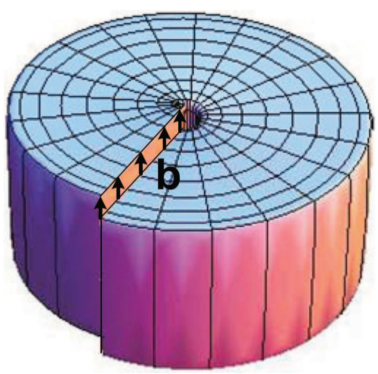

Figure 2: Close the open lips in such a way that a screw dislocation is created along the center line. The resulting medium is what we call the field of a dispiration.